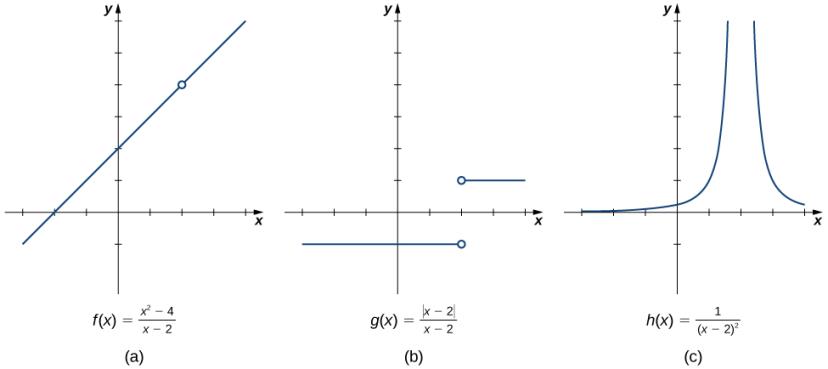
# Section 2.2: The Limit of a Function

To understand the concept of a limit, first look at graphs of functions. For example, looking at the three functions below, we can see that at , all these functions are undefined.



Yet, simply stating they are undefined does not give an accurate picture of what is happening around .

## Intuitive Definition of a Limit

Looking at the graphs above, we see that the behavior of the function as approaches can be very different depending on the function.

Let be a function defined at all values in an open interval containing , except maybe at itself, and let be a real number. If all values of the function approach the real number as the values of approach the number , then the **limit of as approaches is** . Symbolically,

.

In other words, as gets closer to , gets closer to .

For example, in the first function , as approaches from either side, the values of approach . So, mathematically,

.

One way to approximate a limit is to use a table. Choose sets of values – one set approaching from the left (values slightly smaller than ) and another set approaching from the right (values slightly larger than ).

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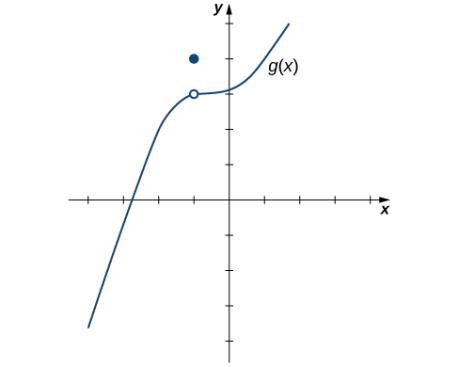
Then, look at each of the columns, determine whether the outputs seem to be approaching a single value. If both columns approach a common value , then we say

Media: Watch this [video](https://www.youtube.com/watch?v=l7Tcay720vw) example on estimating a limit with a table.

Media: Watch this [video](https://youtu.be/Vi4BiJj-n0g) example on finding limits from a graph.

Examples:

1. Evaluate each of the following limits using a table of function values. Then use a graph to confirm your estimate.
2. For shown below, evaluate .



## The Existence of a Limit

For a limit of a function to exist at a point, the function values must approach a single real-number value at that point. If the function values do not approach a single value, then the limit does not exist.

Media: Watch this [video](https://youtu.be/58u7vaqHg68) example on finding the limit of the sine function with a table.

Example: Evaluate using a table of values.

## One-Sided Limits

Indicating that the limit of a function fails to exist at a point does not always provide us with enough information about the behavior of the function at that point. Instead, we look at what happens as we approach the point from the left and right sides.

We define two types of **one-sided limits**.

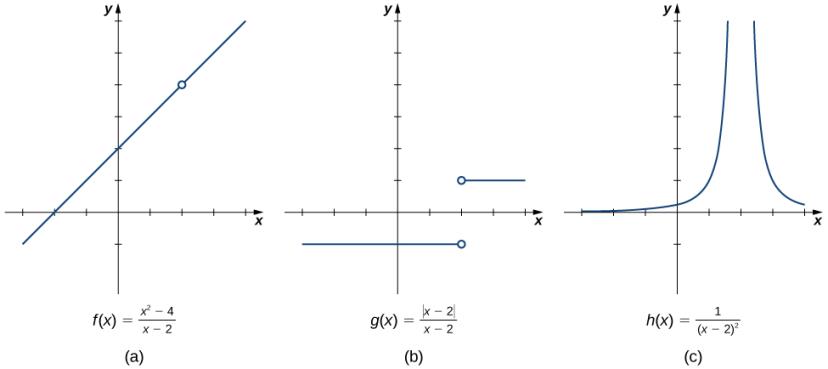
**Limit from the left:** Let be a function defined at all values in an open interval of the form , and let be a real number. If the values of the function approach the real number as the values of (where ) approach the number , then we say that is the limit of as approaches from the left. Symbolically, we express this idea as

.

**Limit from the right:** Let be a function defined at all values in an open interval of the form , and let be a real number. If the values of the function approach the real number as the values of (where ) approach the number , then we say that is the limit of as approaches from the right. Symbolically, we express this idea as

.

So, looking back at the three graphs from the beginning, we can see that the values of the second graph approach different values as we approach from the left and right.



In this case, the since as approaches 2 from the left, (values) approach .

Similarly, the since as since as approaches 2 from the right, (values) approach 1.

So, if the limit from the right and the limit from the left have a common value, then that common value is the limit of the function at that point. If the limit from the left and the limit from the right take on different values, the limit of the function does not exist at that point.

**Relating One-Sided and Two-Sided Limits**

Let be a function defined at all values in an open interval containing , with the possible exception of itself, and let be a real number. Then,

if and only if and .

Media: Watch this [video](https://youtu.be/9geXUOXTJJQ) example on finding one-sided limits from a table.

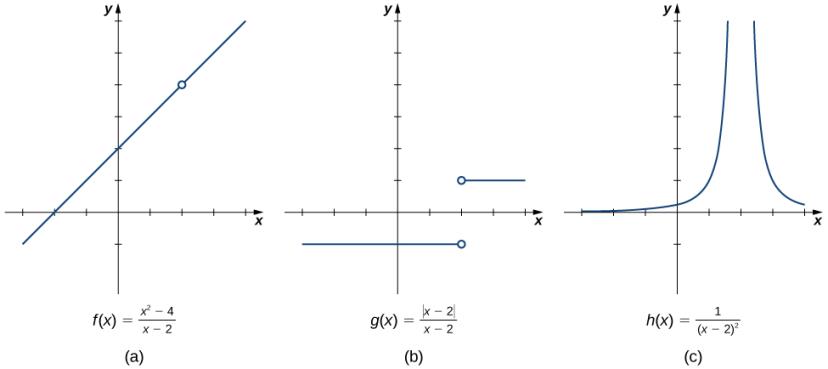
Examples

1. Use a table of values to estimate the following limits, if possible.
2. For the function evaluate each of the following limits.

## Infinite Limits

Evaluating the limit (or right and left limit) of a function at a point helps us to characterize the behavior of a function around a given value. We can also describe the behavior of functions that do not have finite limits.

Looking back at the original three functions, we see that as approaches , the values of becomes larger and larger. Hence, we say . This is called an **infinite limit**.



**Infinite limits from the left:** Let be a function defined at all values in an open interval of the form .

If the values of increase without bound as the values of (where ) approach the number , then we say that the limit as approaches from the left is positive infinity and we write

If the values of decrease without bound as the values of (where ) approach the number , then we say that the limit as approaches from the left is negative infinity and we write

**Infinite limits from the right**: Let be a function defined at all values in an open interval of the form .

If the values of increase without bound as the values of (where ) approach the number , then we say that the limit as approaches from the right is positive infinity and we write

If the values of decrease without bound as the values of (where ) approach the number , then we say that the limit as approaches from the right is negative infinity and we write

**Two-sided infinite limit:** Let be defined for all in an open interval containing .

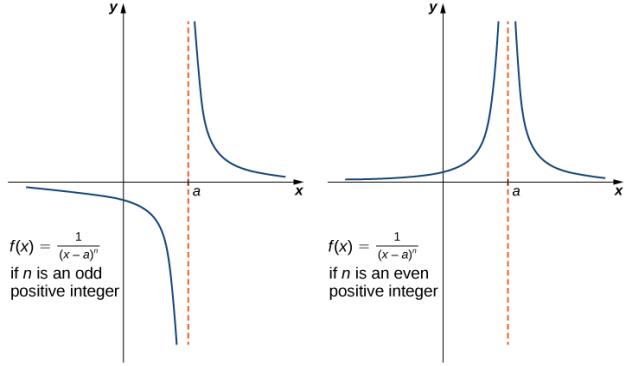
If the values of increase without bound as the values of (where ) approach the number , then we say that the limit as approaches is positive infinity and we write

If the values of decrease without bound as the values of (where ) approach the number , then we say that the limit as approaches is negative infinity and we write

\*Note, when we write statements such as or , we are NOT stating that the limit exists. We are simply describing the behavior of the function.

Examples: Evaluate each of the following limits, if possible. Use a table of values and graph to confirm your conclusion.

Functions of the form where is a positive integer, have infinite limits as approaches from either the left or the right.



If is a positive even integer, then

.

If is a positive odd integer, then

and

.

Notice, that for these types of graphs, we also have vertical asymptotes at . We can determine whether a function has vertical asymptotes by looking at limits.

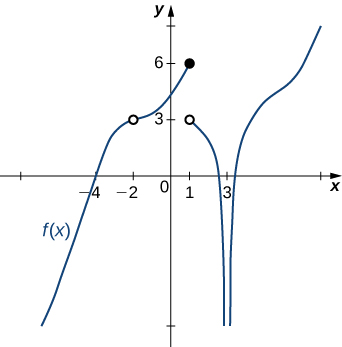
Let be a function. The line is a **vertical asymptote** of , if any of the following conditions hold:

Media: Watch these [video](https://www.youtube.com/watch?v=3YFuOwcog6k)1 and [video 2](https://www.youtube.com/watch?v=UkjgJQaGx98) examples on finding limits graphically.

Media: Watch this [video](https://youtu.be/LJCLiCOr4ek) example on drawing a graph given specific limit properties.

Examples

1. Use the graph of shown below to determine each of the following:



1. ; ; ;
2. ; ; ;
3. ; ; ;
4. ; ; ;
5. Evaluate each of the following limits. Identify any vertical asymptotes of the function
6. Sketch the graph of a function with the given properties:
   * As
   * As